Sections 8.2-8.4 part 2

Chapter 8 Normal Subgroups

<u>Def</u> A subgroup $N \subset G$ such that $Na = aN$ for every $a \in G$ is called <u>normal</u>.

$$G = \bigcup_{a\in G} Na = \bigcup_{a \in G} aN$$ left and right cosets are the same

The set of cosets is denoted by G/N } In ring theory, that was R/I for an ideal I

Group structure - operation - on G/N

<u>Def</u> $(Na)\cdot(Nc) = Nac$

(the coset whose representative is a)(the coset whose representative is c)

= (the coset whose representative is ac)

Th 8.10, 8.12 This operation is well-defined:

For any $b \in Na$ and $d \in Nc$, we have $Nbd = Nac$

<u>Pf</u> Since the cosets are equivalence classes, it suffices to check <u>$bd \in Nac$</u>

$b \in Na$ means $b = n_1 a$ with $n_1 \in N$

$d \in Nc$ means $d = n_2 c$ with $n_2 \in N$

$bd = n_1 a n_2 c$ It may be that $an_2 \neq n_2 a$

However, $aN = Na$ - the subgroup N is normal

$an_2 \in aN = Na$ means $an_2 = n_3 a$ | such $n_3 \in N$ exists

$bd = n_1 a n_2 c = n_1 n_3 a c \in Nac$ because $n_1 n_3 \in N$ as soon as both $n_1 \in N$, $n_3 \in N$

Th 8,13 (1) G/N with this operation is a group

Pf - one checks with group axioms } the identity is the coset of the identity in G

$N = Ne_G = e_G N$

Terminology

G/N is called factor-group
quotient group

Characterizations of normality

Th 8.11 A subgroup $N \subset G$ of a group G is normal if and only if:

(2) $a^{-1}Na \subseteq N$ for every $a \in G$ $\quad a^{-1}Na = \{a^{-1}na \mid n \in N\}$

(3) $aNa^{-1} \subseteq N$ —— " ——

(4) $a^{-1}Na = N$ —— " ——

(5) $aNa^{-1} = N$